FLOW PAST A THIN AXISYMMETRIC BODY WITH A GIVEN VELOCITY DISTRIBUTION ALONG PART OF ITS SURFACE

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Longitudinal flow past a thin body of revolution, part of whose surface is not known a priori and is to be determined from the tangential velocity specified there (free-flow boundary), is considered. The flow is assumed to be vortex-free, and the fluid to be ideal and incompressible. An integral equation for the form of the free surface is derived and is solved by the method of successive approximations. Conditions for the existence and uniqueness of the solution are given. A constant velocity flow along the free boundary (cavitation flow) is considered as a particular example of the general theory.

§1. Statement of the problem. Let a stream with the velocity U_∞ at infinity flow past an axisymmetric body. We assume the form of the body surface to be specified at the leading and trailing edges only, while that of its intermediate part is unknown—here the tangential-velocity distribution is assigned (Fig. 1).

The problem reduces to finding the potential φ of the flow past the body of revolution defined above.

We introduce the following notation: the known parts of the body surface are called "fixed boundaries" (segments AB and CD in Fig. 1), and their equations are expressed by 6K41 $\rho = r_\pm(z)$; the unknown part of the surface is referred to as the "free boundary" (segment BC) with its unknown equation written as $\rho = r(z)$. Thus the complete streamlined surface $\rho = R(z)$ will be defined piecewise by

$$R(z) = r_-(z) \quad (a < z < b), \qquad R(z) = r(z) \quad (b < z < c),$$
$$R(z) = r_+(z), \quad (c < z < d). \tag{1.1}$$

In the following all parameters related to sections (a, b) and (c, d) will be denoted, as in (1.1), by minus and plus subscripts.

The problem stated reduces to solution of the equation

$$\varphi_{zz} + \varphi_{\rho\rho} + \rho^{-1}\varphi_\rho = 0$$

for the external surface $\rho = R(z)$ with the boundary values

$$\partial\varphi / \partial n = 0 \text{ for } \rho = R(z),$$
$$\varphi_z \to u_\infty z \quad \text{as} \quad \rho \to \infty$$

and the proviso that the function $R(z)$ is unknown along section (b, c), where the tangential velocity $d\varphi/d\tau = V_\tau(Z)$ is specified.

§2. Conditions of the problem. We assume the body to be thin hence:

1) the functions $R(z)$ and $R'(z)$ are continuous, and $R(a) = R(d) = 0$

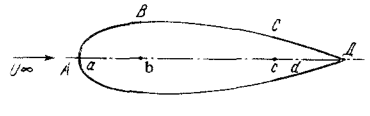

Fig. 1

2) the function $R^2(z)$ is analytic in the neighborhood of critical points a and d

3) the function $R''(z)$ is piecewise-continuous.

For simplicity we assume that discontinuities are allowable only at points b and c (the points of contact of the "fixed" and the "free" boundaries).

4) all along $[a, d]$ the following thinness conditions are fulfilled

$$R^2 / (d-a)^2 < \varepsilon, \quad |(R^2)^{(k)}| / (d-a)^{2-k} < \varepsilon$$
$$(k = 1, 2, 3).$$

Condition (2) means that in the neighborhood of critical points an expansion of the form

$$R^2(z) = z^n (a_0 + a_1 z + \ldots)$$

is valid.

At a critical point the streamlined surface behaves for $n = 1$ as a sphere, for $n = 2$ as a cone, and for $n \geq 3$ as a knife-edge.

Condition (4) depends on the dimensions of the body. For a finite length $d - a = 1$ can be assumed, and these inequalities are then written thus:

$$R^2 < \varepsilon, \quad |(R^2)^{(k)}| < \varepsilon \quad (k = 1, 2, 3). \tag{2.1}$$

The thinness conditions for infinite caverns are written in the form

$$|(R^2)''| < \varepsilon, \qquad |(R^2)'''| < C < \infty. \tag{2.2}$$

Particular attention will be given in this article to a body of finite length, so that $a = 0$ and $d = 1$ can be assumed.

The velocity V_τ at the free boundary must fulfill the relation

$$\left| \frac{V_\tau - u_\infty}{u_\infty} \right| < \frac{1}{4} \varepsilon |\ln \varepsilon|. \tag{2.3}$$

Let us assume that $R'^2 < \varepsilon$. Then from the equation

$$V_\tau = \sqrt{1 + R'^2}\, \partial\varphi/\partial z.$$

it follows that $d\varphi/dz = V_\tau$ can be assumed for small ε, and, consequently, the value of the potential

$$\varphi = \varphi_0 + \int_b^z V_\tau(z)\, dz. \tag{2.4}$$

along the free boundary is known.

Here φ_0 is a certain constant. Specification of the velocity on the surface of a thin body of revolution is thus tantamount to our defining the flow potential there. It was shown in [1] that for small ε the potential of longitudinal flow past a thin body is given the formula

$$\varphi = u_\infty \left\{ z + \frac{1}{4} \int_0^1 P(\zeta) \frac{z-\zeta}{[(z-\zeta)^2+\rho^2]^{3/2}} d\zeta + \frac{1}{4} \Gamma(z, \rho) \right\}, \quad (2.5)$$

$$P(\zeta) = R^2(\zeta)[1-\gamma(\zeta)], \quad (2.6)$$

$$\Gamma(z, \rho) = 2 \sum_{i=1}^{2} \rho_i^3 \frac{z-c_i}{[(z-c_i)^2+\rho^2]^{3/2}},$$

$$\gamma(\zeta) = \sum_{i=1}^{2} \frac{\rho_i^3}{[(\zeta-c_i)^2+R^2(\zeta)]^{3/2}},$$

$$P(\zeta) = p_-(\zeta) \ (0<\zeta<b), \quad P(\zeta) = p(\zeta) \ (b<\zeta<c),$$

$$P(\zeta) = p_+(\zeta) \ (c<\zeta<1). \quad (2.7)$$

Here $c_1 = \rho_1$, $c_2 = 1 - \rho_2$, and ρ_1 and ρ_2 are the ratii of body curvature at critical points.

It follows from formula (2.5) that the flow potential can be readily expressed in terms of the equation for the streamlined surface R(z). Consequently, for solution of this problem it is sufficient to find the equation for the free surface R(z). Since R(z) appears in (2.5) in terms of P(z) defined by (2.6), the problem essentially reduces to our finding the function P.

We note that outside of the neighborhood of critical points the function $\gamma(z)$ is on the order of ε^3, hence almost everywhere $P(z) \approx R^2(z)$.

§3. Relation to boundary-value problems. The vortex-free flow of an ideal incompressible fluid past a solid with the surface S reduces to the Neumann problem—the derivation of a solution of the Laplace equation $\Delta\varphi\,\Delta\varphi = 0$ with boundary condition $d\varphi/dn \; s = 0$. Here the surface S is assumed to be given.

In our problem part of the surface is unknown, but the Dirichlet condition (2.4) is specified there. Hence, we have to solve the Neumann problem in which part of the boundary is unknown and is determined on the basis of the Dirichlet condition. The solvability of this problem will be dealt with later.

§4. The fundamental integral equation. Let us consider Eq. (2.5) for $z \in (b, c)$ and $\rho = r(z)$. By substituting expression (2.4) for the potential on the left we can obtain the relationship

$$\int_0^1 P(\zeta) K_\zeta(\zeta, z, r) d\zeta - S(z) = 0, \quad (4.1)$$

$$K_\zeta(\zeta, z, r) = \frac{z-\zeta}{[(z-\zeta)^2+r^2(z)]^{3/2}} - \frac{b-\zeta}{[(b-\zeta)^2+r^2(b)]^{3/2}}, \quad (4.2)$$

$$S(z) = \frac{4}{u_\infty}\left[\int_b^z V_\tau dz - u_\infty(z-b)\right] - \Delta\Gamma(z, r), \quad (4.3)$$

$$\Delta\Gamma(z, r) = \Gamma(z, r(z)) - \Gamma(b, r(b)). \quad (4.4)$$

Thus the unknown function $P(\xi)$ ($\xi \in (b, c)$ is a solution of the nonlinear integral equation (4.1) with constant limits of integration. This relationship can be rewritten in the form

$$\int_0^1 P'(\zeta) K(\zeta, z, z) d\zeta + S(z) = 0$$

$$K(\zeta, z, r) = \frac{1}{\sqrt{(\zeta-z)^2+r^2(z)}} - \frac{1}{\sqrt{(\zeta-b)^2+r^2(b)}} \quad (4.5)$$

Relationship (4.5) is the fundamental equation for this problem. When rewritten in the form

$$\int_b^c p'(\zeta) K(\zeta, z, r) d\zeta =$$

$$= -S(z) - \int_0^b p'_-(\zeta) K(\zeta, z, r) d\zeta - \int_c^1 p_+'(\zeta) K(\zeta, z, r) d\zeta$$

with the nonlinearity of kernel K neglected and its right-hand side considered as known, it reduces to a Fredholm integral equation of the first kind with respect to function $p'(\zeta)$.

§5. Solution of the integral equation. Let us assume

$$\alpha = 1 / |\ln r^2(b)|. \quad (5.1)$$

With the equality

$$\int_{-A}^{A} \frac{dx}{\sqrt{x^2+r^2}} = -\ln r^2 + \ln(A + \sqrt{A^2+r^2})^2$$

it can be proved that

$$\alpha \int_0^1 P'(\zeta) K(\zeta, z, r) d\zeta \to P'(z) - P'(b) \quad (5.2)$$

$$\text{as } \varepsilon \to 0.$$

On the basis of this relationship we write the fundamental equation (4.5) in the form

$$\tau[P] \equiv \pi[P] + T[P] = 0, \quad (5.3)$$

$$\pi[P] = P'(z) - P'(b) + \alpha S(z),$$

$$T[P] = \alpha \int_0^1 P'(\zeta) K(\zeta, z, r) d\zeta - [P'(z) - P'(b)],$$

Here $\pi[P]$ is the main part of operator τ, while T[P] is its "supplementary" operator which absorbs the basic complexity of the operator under consideration and in accordance with (5.2) $T[P] \to 0$ as $\varepsilon \to 0$.

To solve Eq. (5.3) we use a modified Newton-Kantorovich method for solution of the functional equations [2]

$$P_{n+1} = P_n - [\pi'[P_1]]^{-1}[\tau[P_n]].$$

Here $\pi'[P_1]$ is the Frechet derivative of the operator π; $[\pi'[P_1]]^{-1}$ is the inverse operation for π'; P_1 is the solution of the equation $\pi[P_1] = 0$.

In this case (the first approximation)

$$P_1(z) = P(b) + P'(b)(z-b) - \alpha \int_b^z S(z) dz, \quad (5.5)$$

$$[\pi'[P_1]]^{-1}[\tau] = \int_b^z \tau dz. \quad (5.6)$$

Thus the method of successive approximations for solving Eq. (4.5) is

$$P_{n+1} = P_n - \alpha \int_b^z \left\{ \int_0^1 P_n'(\zeta) K(\zeta, z, r_n) d\zeta + S(z) \right\} dz, \quad (5.7)$$

and function (5.5) can be taken as the first approximation.

We note that the function $P_n(\xi)$ is defined by

$$P_n(\zeta) = p_-(\zeta) \ (0<\zeta<b), \quad P_n(\zeta) = p_n(\zeta) \ (b<\zeta<c),$$

$$P_n(\zeta) = p_+(\zeta) \quad (c<\zeta<1,$$

which follows from expression (1.1).

§ 6. Existence and uniqueness of the solution of the equation. The first two Frechet derivatives of the operator T[P] are

$$T'[P] = \alpha \int_b^c \delta p'(\zeta) K(\zeta, z, r)\, d\zeta - \delta p' -$$

$$-\frac{1}{2}\alpha\delta p \int_0^1 \frac{P'(\zeta)}{[(\zeta-z)^2+r^2]^{3/2}} d\zeta, \quad (6.1)$$

$$T''[P] = -\alpha\delta p \int_b^c \frac{\delta p'(\zeta)}{[(\zeta-z)^2+r^2]^{3/2}} d\zeta +$$

$$+\frac{3}{4}\alpha(\delta p)^2 \int_0^1 \frac{P'(\zeta)}{[(\zeta-z)^2+r^2]^{5/2}} d\zeta. \quad (6.2)$$

$$|T[P_1]| < \frac{1}{|\ln\varepsilon|}\left\{|P_1'|\left|\ln\frac{\mu\varepsilon}{r_1^2}\right| + 4\max|P_1''|\right\},$$

$$|T'[P_1]| < \frac{1}{|\ln\varepsilon|}\left\{\left|\ln\frac{\mu\varepsilon}{r_1^2}\right| + 4A_0 + \frac{\max|P_1''|}{r_1^2}\right\},$$

$$|T''[P_1]| < \frac{1}{|\ln\varepsilon|}\left\{\frac{1}{r_1^2} + \frac{3}{2}\frac{\max|P_1'|}{r_1^4}\right\}.$$

To prove this lemma it is necessary to use the expressions for T[P] from (5.3), (6.1), and (6.2).

For simplicity we assume here that $\delta p = \delta r^2$ and that the increments of functionals for both differentiations are equal.

Lemma 1. Let $r^2(b) = \mu\varepsilon$ $(0<\mu<1)$ and $|\delta p''| < A_0$. Then for $|\delta p| = |\delta p'| = 1$ the following expressions for T[P] from (5.3), (6.1), and (6.2).

Corollary 1. For the estimates of functionals to be bounded it is sufficient to set

$$r_1^2(z) > m\varepsilon > 0. \quad (6.3)$$

Corollary 2. Let the function $r_1^2(z)$ (as well as $P_1(z)$) satisfy conditions (2.1) for a thin body. The estimates of operators are then written in the form

$$|T[P_1]| < \frac{\varepsilon}{|\ln\varepsilon|}C_1, \quad |T'[P_1]| < \frac{1}{\ln\varepsilon}C_2,$$

$$|T''[P_1]| < \frac{C_3}{\varepsilon|\ln\varepsilon|}, \quad C_1 = \max\left|\ln\frac{\mu\varepsilon}{r_1^2}\right| + 4, \quad (6.4)$$

$$C_2 = \max\left|\ln\frac{\mu\varepsilon}{r_1^2}\right| + 4A_0 + \frac{1}{m}, \quad C_3 = \frac{4}{m}.$$

It remains to determine the conditions in which $P_1(z)$ in (5.5) satisfies the limitations for a thin body.

Lemma 2. If the function $P_1(z)$ is to satisfy the conditions for a thin body, it is sufficient to require the determination of velocity V_τ by inequality (2.3) along the free boundary.

Proof. The condition $|(R^2)''| < \varepsilon$ for a thin body is in this case of the form

$$4\frac{1}{|\ln r^2(b)|}\left|\frac{V_\tau - u_\infty}{u_\infty}\right| + \left|\frac{d}{dz}\Gamma(z,r)\right| < \varepsilon$$

$$\left(\left|\frac{d}{dz}\Gamma(z,r)\right| = O(\varepsilon^3)\right).$$

This proves the lemma.

Let us consider the class of differentiable functions with bounded second derviatives. We introduce a norm of the form $\|P\| = \max(|P| + |P'|)$ which the following estimate of the functional $[\pi'[P_1]]^{-1}[F]$ in (5.7) is valid

$$\|[\pi'[P_1]]^{-1}[F]\| < 2\max|F|. \quad (6.5)$$

On these assumptions the basic result is valid.

Theorem. Let the following conditions for the flow past an axisymmetric body to be fulfilled:

(a) length c − b of the free boundary satisfies the inequality $c - b < z_0 - b$, where z_0 is that root of equation $P_1(z_0) = 0$ nearest to b,

(b) functions $p_-(z)$ and $p_+(z)$ satisfy the limitations of a thin body (2.1),

(c) at the free boundary velocity V_τ is continuous and satisfies (2.3).

The process (5.8) of successive approximations reduces for rather small ε to solution of Eq. (4.5), and this solution is unique.

Proof. Condition (a) implies the fulfillment of inequality (6.3), with $m\varepsilon = \{P_1(z)[1-\gamma]\}$, hence the functionals of lemma 1 are bounded. Conditions (b) and (c) coincide with those of lemma 2, hence estimates (6.4) hold for the functionals.

To prove convergence we have to establish that the inequalities [2]

$$\|[\pi'[P_1]]^{-1}[T[P_1]]\| = \eta < 1/2,$$

$$\|[\pi'[P_1]]^{-1}[T''[P_1]]\| = M < \infty$$

$$\|[\pi'[P_1]]^{-1}[T'[P_1]]\| = \mu < 1,$$

$$\|[\pi'[P_1]]^{-1}[\pi''[P_1]]\| = K < \infty$$

are satisfied, provided that

$$h = \frac{\eta(M+K)}{(1-\mu)^2} < \frac{1}{2}, \qquad \frac{1-\sqrt{1-2\pi}}{h}\frac{\eta}{1-\mu} < \varepsilon,$$

$$\frac{1+\sqrt{1-2h}}{h}\frac{\eta}{1-\mu} > \varepsilon.$$

Yet according to relationship (6.5) the estimates of the first four functionals are

$$\eta < 2C_1\varepsilon/|\ln\varepsilon|, \quad \mu < 2C_2/|\ln\varepsilon|,$$

$$M < 2C_3/\varepsilon|\ln\varepsilon|, \quad K = 0,$$

and, consequently, the complete system of inequalities can be satisfied for rather small ε. The theorem is proved.

§7. Conditions of solvability of the problem stated. The requirement for R(z) and R'(z) to be continuous leads (in the context of thin-body theory to the following conditions at the contact points of the boundaries:

$$r(b) = r_-(b), \quad r'(b) = r_-(b),$$

$$r(c) = r_+(c), \quad r_+'(c) = r'(c).$$

The first two relationships concerning continuity at point b, have already been satisfied in construction of the function $P_1(z)$ (5.5) and it remains only to fulfill the conditions of continuity at the point c

$$p(c) = p_+(c), \qquad p'(c) = p_+'(c). \quad (7.1)$$

This means that restrictions are to be imposed on two numerical parameters of flow (free parameters), if this problem is to be solvable. For example, the case of cavitation flow (V_τ = const) the number Q of cavitations and the cavern length c − b, etc. can be taken as such parameters.

Should the conditions of the problem require continuity of R''(z) at the point b, the number of free parameters would be increased by one. We note that in this problem the Neumann condition is automatically fulfilled, since $\partial\varphi/\partial n_s = 0$.

§8. Solution of the problem stated. This section gives a summary of the results obtained.

The free boundary equation r(z) is defined by the equality $p(z) = r^2(1-\gamma(z))$ in (2.6), where the function p(z) is found by the method of successive approximations

$$P_{n+1}(z) = P_n(z) - \alpha\int_b^z\left\{\int_0^1 P_n'(\zeta)K(\zeta,z,r_n)\,d\zeta + S(z)\right\}dz,$$

$$\alpha = \frac{1}{|\ln r_-^2(b)|}$$

$$P_n(\zeta) = \begin{cases} r_-^2(\zeta)[1-\gamma(\zeta)] & (0<\zeta<b) \\ r_n^2(\zeta)[1-\gamma(\zeta)] & (b<\zeta<c) \\ r_+^2(\zeta)[1-\gamma(\zeta)] & (c<\zeta<1), \end{cases}$$

$$K(\zeta, z, r) = \frac{1}{\sqrt{(\zeta - z)^2 + r^2(z)}} - \frac{1}{\sqrt{(\zeta - b)^2 + r^2(b)}},$$

$$S(z) = \frac{u}{u_\infty}\left[\int_b^z V_\tau dz - u_\infty (z - b)\right] - \Delta\Gamma(z, r),$$

$\gamma(z)$ and $\Delta\Gamma(z, r)$ are determined from formulas (2.7) and (4.4).

Simultaneously with derivation of the form of the free boundary it is necessary to determine the free parameters from the condition

$$p(c) = p_+(c), \qquad p'(c) = p_+'(c).$$

Should the conditions of the problem require continuity of $R''(z)$, the number of free parameters would be correspondingly increased.

Note. The quantity $\gamma(z)$ is actually different from zero only in the neighborhood of critical points, and away from them it is on the order of ε^3. For example, for an ellipsoid of revolution with a ratio of axes of 1/3 we have $R^2(z) < 1/36$ we have $\gamma = O(10^{-4})$ almost everywhere.

§9. Cavitation flow. Let us consider, as an example, the case of $V_\tau = U_0$. The first approximation (5.5) (away from the critical points) is written in the form

$$r^2(z) = p_-(b) + p_-'(b)(z - b) - \frac{h_0}{|\ln r^2(b)|}(z - b)^2, \qquad h_0 = 2\left(\frac{u_0}{u_\infty} - 1\right). \tag{8.1}$$

It will be seen that the free line of flow has the form of an elongated ellipse with $(h_0 / |\ln r^2|)^{1/2}$ as the ratio of its axes. This qualitative result can be verified directly, since on the surface of an ellipsoid $\varphi = z$ const [3], i.e., φ_z const.

When the point C recedes into infinity, the ellipse degenerates into a parabola

$$r(z) = \sqrt{p_-(b) + p_-'(b)(z - b)}.$$

This result of the first approximation coincides with the asymptotic law $r(z) = O(z^{1/2}(\ln z)^{-1/2})$ cited in [4].

The quantity h_0 in (8.1) is readily expressed in terms of the number of cavitations

$$h_0 = 2(\sqrt{1 + Q} - 1).$$

For a small number of cavitations $h_0 = Q$.

If the cavitation flow is symmetric with respect to plane $z = \text{const}$, (the Ryabushinskii flow) system (7.1) reduces to the single equation $p'(c) = p_+'(c)$.

Thus, in the case of Ryabushinskii cavitation flow a restriction must be imposed on one parameter of the problem. For a given form of the fixed boundary this restriction can be considered as an implicit dependence of length $b - c$ of a theoretical cavern on the number Q of cavitations.

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